

Sums of two triangularizable quadratic matrices over an arbitrary field

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Abstract

Let \mathbb{K} be an arbitrary field, and a, b, c, d be elements of \mathbb{K} such that the polynomials $t^2 - at - b$ and $t^2 - ct - d$ are split in $\mathbb{K}[t]$. Given a square matrix $M \in M_n(\mathbb{K})$, we give necessary and sufficient conditions for the existence of two matrices A and B such that $M = A + B$, $A^2 = aA + bI_n$ and $B^2 = cB + dI_n$. Prior to this paper, such conditions were known in the case $b = d = 0$, $a \neq 0$ and $c \neq 0$ [4] and in the case $a = b = c = d = 0$ [1]. Here, we complete the study, which essentially amounts to determining when a matrix is the sum of an idempotent and a square-zero matrix. This generalizes results of Wang [5] to an arbitrary field, possibly of characteristic 2.

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1 Introduction

1.1 Basic notations and aims

Let \mathbb{K} be an arbitrary field, and $\overline{\mathbb{K}}$ an algebraic closure of it. We denote by $\text{car}(\mathbb{K})$ the characteristic of \mathbb{K} . We denote by $M_n(\mathbb{K})$ the algebra of square matrices with

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n rows and entries in \mathbb{K} , and by I_n its identity matrix. Similarity of two square matrices A and B is denoted by $A \sim B$. Given $M \in M_n(\mathbb{K})$, we denote by $\text{Sp}(M)$ the set of eigenvalues of M in the field \mathbb{K} . We denote by \mathbb{N} the set of non-negative integers, and by \mathbb{N}^* the set of positive ones.

A matrix of $M_n(\mathbb{K})$ is called **quadratic** when it is annihilated by a polynomial of degree two. More precisely, given a pair $(a, b) \in \mathbb{K}^2$, a matrix A of $M_n(\mathbb{K})$ is called **(a, b) -quadratic** when $A^2 = aA + bI_n$. In particular, a matrix is $(1, 0)$ -quadratic if and only if it is idempotent, and it is $(0, 0)$ -quadratic if and only if it is square-zero.

Let $(a, b, c, d) \in \mathbb{K}^4$. A matrix is called an **(a, b, c, d) -quadratic sum** when it may be decomposed as the sum of an (a, b) -quadratic matrix and of a (c, d) -quadratic one. Note that a matrix which is similar to an (a, b, c, d) -quadratic sum is an (a, b, c, d) -quadratic sum itself. Our aim here is to give necessary and sufficient conditions for a matrix of $M_n(\mathbb{K})$ to be an (a, b, c, d) -quadratic sum. In [5], Wang has expressed such conditions in terms of rational canonical forms when \mathbb{K} is the field of complex numbers, and his proof actually encompasses the more general case of an algebraically closed field of characteristic not 2. In our recent [4], we have worked out the case $b = d = 0$, $a \neq 0$ and $c \neq 0$, i.e., we have determined when a matrix may be written as $aP + cQ$, where P and Q are idempotent matrices (this generalized earlier results of Hartwig and Putcha [3]). In [1], Botha has worked out the case $a = b = c = d = 0$ for an arbitrary field, generalizing results of Wang and Wu [6]; as in [4], fields of characteristic 2 yield somewhat different results than the others.

The purpose of this paper is to solve the remaining cases, assuming that the polynomials $t^2 - at - b$ and $t^2 - ct - d$ are split over \mathbb{K} .

The basic strategy is to reduce the situation to a more elementary one. Assume, for the rest of the section, that $t^2 - at - b$ and $t^2 - ct - d$ are split over \mathbb{K} , and let α be a root of $t^2 - at - b$ and β be one of $t^2 - ct - d$. Then an (a, b) -quadratic matrix is a matrix of the form $\alpha I_n + P$, where P is $(a - 2\alpha, 0)$ -quadratic. We deduce that a matrix of $M_n(\mathbb{K})$ is an (a, b, c, d) -quadratic sum if and only if it splits as $(\alpha + \beta).I_n + M$, where M is an $(a - 2\alpha, 0, c - 2\beta, 0)$ -quadratic sum.

We are thus reduced to studying the case $b = d = 0$.

In the case $b = d = 0$ and $a \neq 0$, notice furthermore that an (a, b, c, d) -quadratic sum is simply the product of a with a $(1, 0, \frac{c}{a}, 0)$ -quadratic sum. Therefore, the case $b = d = 0$ is essentially reduced to three cases:

- (i) $b = d = 0$, $a \neq 0$ and $c \neq 0$;
- (ii) $a = b = c = d = 0$;
- (iii) $a = 1$ and $b = c = d = 0$.

Case (i) has been dealt with in [4], and case (ii) more recently in [1]. Therefore, only case (iii) remains to be studied in order to complete the case where both polynomials $t^2 - at - b$ and $t^2 - ct - d$ are split over \mathbb{K} . In other words, it remains to determine which matrices may be decomposed as the sum of an idempotent and a square-zero matrix. This has been done by Wang in [5] for the case $\mathbb{K} = \mathbb{C}$. Our aim is to generalize his results.

1.2 Main theorem

Definition 1. Let $(u_n)_{n \geq 1}$ and $(v_n)_{n \geq 1}$ be two non-increasing sequences of non-negative integers. Let $p > 0$ be a positive integer. We say that (u_n) and (v_n) are **p -intertwined** when

$$\forall n \geq 1, u_{n+p} \leq v_n \quad \text{and} \quad v_{n+p} \leq u_n.$$

Notation 2. Given $A \in M_n(\mathbb{K})$, $\lambda \in \overline{\mathbb{K}}$ and $k \in \mathbb{N}^*$, we set

$$n_k(A, \lambda) := \dim \text{Ker}(A - \lambda I_n)^k - \dim \text{Ker}(A - \lambda I_n)^{k-1},$$

and

$$j_k(A, \lambda) := n_k(A, \lambda) - n_{k+1}(A, \lambda)$$

i.e., $n_k(A, \lambda)$ (respectively, $j_k(A, \lambda)$) is the number of blocks of size k or more (respectively, of size k) associated to the eigenvalue λ in the Jordan reduction of A .

Our main theorem follows.

Theorem 1. *Let $M \in M_n(\mathbb{K})$. The following conditions are equivalent:*

- (i) M is a $(1, 0, 0, 0)$ -quadratic sum.
- (ii) $\forall \lambda \in \overline{\mathbb{K}} \setminus \{0, 1\}$, $\forall k \in \mathbb{N}^*$, $j_k(M, \lambda) = j_k(M, 1 - \lambda)$, the sequences $(n_k(M, 0))_{k \geq 1}$ and $(n_k(M, 1))_{k \geq 1}$ are 2-intertwined, and, if $\text{car}(\mathbb{K}) \neq 2$, the Jordan blocks of M for the eigenvalue $\frac{1}{2}$ are all even-sized.

- (iii) *There are matrices $A \in M_p(\mathbb{K})$ and $B \in M_{n-p}(\mathbb{K})$ such that $M \sim A \oplus B$, where all the invariant factors of A are polynomials of $t(t-1)$ and A has no eigenvalue in $\{0, 1\}$, the matrix B is triangularizable with $\text{Sp}(B) \subset \{0, 1\}$, and the sequences $(n_k(B, 0))_{k \geq 1}$ and $(n_k(B, 1))_{k \geq 1}$ are 2-intertwined.*

1.3 Structure of the proof

The equivalence between conditions (ii) and (iii) of Theorem 1 is a straightforward consequence of the kernel decomposition theorem and of Proposition 9 of [4], which we restate:

Proposition 2. *Let $A \in M_n(\mathbb{K})$ and $\alpha \in \mathbb{K}$. The following conditions are equivalent:*

- (i) *The invariant factors of A are polynomials of $t(t - \alpha)$.*
- (ii) *For every $\lambda \in \overline{\mathbb{K}}$,*
 - *if $\lambda \neq \alpha - \lambda$, then $\forall k \in \mathbb{N}^*$, $j_k(A, \lambda) = j_k(A, \alpha - \lambda)$;*
 - *if $\lambda = \alpha - \lambda$, then $\forall k \in \mathbb{N}$, $j_{2k+1}(A, \lambda) = 0$.*

The equivalence of (i) and (iii) is much more involving and takes up the rest of the paper:

- In Section 2, we show that the equivalence (i) \Leftrightarrow (iii) needs to be proven only in the following elementary cases:
 - (a) M has no eigenvalue in $\{0, 1\}$;
 - (b) M is triangularizable and $\text{Sp}(M) \subset \{0, 1\}$.
- In Section 3, we prove that (i) \Leftrightarrow (iii) holds in case (a).
- In Section 4, we prove that (i) \Leftrightarrow (iii) holds in case (b).

2 Reduction and reconstruction principles

2.1 A reconstruction principle

Let M_1 and M_2 be two $(1, 0, 0, 0)$ -quadratic sums (respectively in $M_n(\mathbb{K})$ and $M_p(\mathbb{K})$). Split up $M_1 = A_1 + B_1$ and $M_2 = A_2 + B_2$, where A_1, A_2 are idempotent

and B_1, B_2 are square-zero. Then $M_1 \oplus M_2 = (A_1 \oplus A_2) + (B_1 \oplus B_2)$, while $A_1 \oplus A_2$ is idempotent and $B_1 \oplus B_2$ is square-zero. Therefore $M_1 \oplus M_2$ is a $(1, 0, 0, 0)$ -quadratic sum.

2.2 The basic lemma

The following lemma is a key tool to analyze quadratic sums in general.

Lemma 3. *Let $(a, b, c, d) \in \mathbb{K}^4$. Let A and B be respectively an (a, b) -quadratic and a (c, d) -quadratic matrix of $M_n(\mathbb{K})$.*

Then A and B both commute with $(A + B)((a + c)I_n - (A + B))$.

Proof. Set $C := (A + B)((a + c)I_n - (A + B))$ and note that $C = (a + c)(A + B) - A^2 - B^2 - AB - BA = -(b + d)I_n + cA + aB - AB - BA$.

Therefore

$$AC - CA = a(AB - BA) - A^2B + BA^2 = -bB + bB = 0$$

and by symmetry $BC - CB = 0$. □

Corollary 4. *Let $(A, B) \in M_n(\mathbb{K})^2$ such that $A^2 = A$ and $B^2 = 0$. Then A and B both commute with $(A + B)(A + B - I_n)$.*

2.3 Reduction to elementary cases

Let $M \in M_n(\mathbb{K})$. The minimal polynomial μ of M splits up as

$$\mu(t) = P(t) t^p (t - 1)^q,$$

where $P(t)$ has no root in $\{0, 1\}$ and $(p, q) \in \mathbb{N}^2$. Let M_1 (respectively, M_2) be a matrix associated to the endomorphism $X \mapsto MX$ on the vector space $\text{Ker } P(M)$ (respectively, on the vector space $\text{Ker } M^p(M - I_n)^q$). By the kernel decomposition theorem, one has

$$M \sim M_1 \oplus M_2,$$

while $P(M_1) = 0$ and $t^p(t - 1)^q$ annihilates M_2 . If implication (iii) \Rightarrow (i) holds for M_1 and M_2 , then the reconstruction principle of Section 2.1 shows that it also holds for M .

Conversely, assume that $M = A + B$ for a pair $(A, B) \in M_n(\mathbb{K})^2$ with $A^2 = A$ and $B^2 = 0$. By Corollary 4, A and B both commute with $M(M - I_n)$, and

hence they stabilize the subspaces $\text{Im}(M(M - I_n))^n$ and $\text{Ker}(M(M - I_n))^n$ in the Fitting decomposition of $M(M - I_n)$. Using an adapted basis of \mathbb{K}^n for this decomposition, we find $P \in \text{GL}_n(\mathbb{K})$, an integer $p \geq 0$, matrices A_1, B_1 in $\text{M}_p(\mathbb{K})$ and matrices A_2, B_2 in $\text{M}_{n-p}(\mathbb{K})$ such that

$$A = P(A_1 \oplus A_2)P^{-1} \quad \text{and} \quad B = P(B_1 \oplus B_2)P^{-1},$$

the matrices $M_1 := A_1 + B_1$ and $M_2 := A_2 + B_2$ being both $(1, 0, 0, 0)$ -quadratic sums, with $M_1(M_1 - I_p)$ non-singular and $M_2(M_2 - I_{n-p})$ nilpotent. In other words, M_1 has no eigenvalue in $\{0, 1\}$ and M_2 is triangularizable with $\text{Sp}(M_2) \subset \{0, 1\}$. If implication (i) \Rightarrow (iii) holds for both M_1 and M_2 , then it clearly holds for M .

We conclude that equivalence (i) \Leftrightarrow (iii) needs to be proven only in the following special cases:

- (a) M has no eigenvalue in $\{0, 1\}$;
- (b) M is triangularizable with $\text{Sp}(M) \subset \{0, 1\}$.

3 The case M has no eigenvalue in $\{0, 1\}$

3.1 A lemma on companion matrices

Notation 3. Given a monic polynomial $P = t^n - a_{n-1}t^{n-1} - \dots - a_1t - a_0 \in \mathbb{K}[t]$, we denote its *companion matrix* by

$$C(P) := \begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & & 0 & a_1 \\ 0 & 1 & 0 & \dots & 0 & a_2 \\ & & \ddots & \ddots & & \vdots \\ \vdots & & & 1 & 0 & a_{n-2} \\ 0 & \dots & \dots & 0 & 1 & a_{n-1} \end{bmatrix} \in \text{M}_n(\mathbb{K}).$$

Notation 4. For $E \in \text{M}_p(\mathbb{K})$, we set

$$U_E := \begin{bmatrix} I_p & E \\ I_p & 0_p \end{bmatrix} \in \text{M}_{2p}(\mathbb{K}).$$

We start with two easy lemmas on the matrices of type U_E .

Lemma 5. *Given two similar matrices E and E' of $M_p(\mathbb{K})$, the matrices U_E and $U_{E'}$ are similar.*

Proof. Choosing $R \in GL_p(\mathbb{K})$ such that $E' = RER^{-1}$, a straightforward computation shows that

$$U_{E'} = (R \oplus R) U_E (R \oplus R)^{-1}.$$

□

Conjugating by a well-chosen permutation matrix, the following result is straightforward:

Lemma 6. *Given square matrices A and B , one has $U_{A \oplus B} \sim U_A \oplus U_B$.*

We now examine the case E is a companion matrix. The following lemma generalizes Lemma 14 of [4] and is the key to equivalence (i) \Leftrightarrow (iii) in Theorem 1 for a matrix with no eigenvalue in $\{0, 1\}$:

Lemma 7. *Let $(\alpha, \beta) \in \mathbb{K}^2$. Let $P(t)$ be a monic polynomial of degree n . Then*

$$\begin{bmatrix} \alpha I_n & C(P) \\ I_n & \beta I_n \end{bmatrix} \sim C(P((t - \alpha)(t - \beta))).$$

Lemma 7 was stated and proved in [4] with the extra condition that $\alpha \neq 0$ and $\beta \neq 0$, but an inspection of the proof shows that this condition is unnecessary.

Corollary 8. *Let $P \in \mathbb{K}[t]$ be a monic polynomial. Then the companion matrix $C(P(t(t - 1)))$ is a $(1, 0, 0, 0)$ -quadratic sum.*

Proof. Indeed, Lemma 7 shows, with $n := \deg P$, that

$$C(P(t(t - 1))) \sim A + B \quad \text{with} \quad A = \begin{bmatrix} I_n & 0_n \\ I_n & 0_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0_n & C(P) \\ 0_n & 0_n \end{bmatrix}.$$

Obviously, $A^2 = A$ and $B^2 = 0$, and hence $C(P(t(t - 1)))$ is the sum of an idempotent and a square-zero matrix. □

3.2 Application to $(1, 0, 0, 0)$ -quadratic sums

Let $M \in M_n(\mathbb{K})$.

- Assume that each invariant factor of M is a polynomial of $t(t-1)$. Then we may find monic polynomials P_1, \dots, P_p such that

$$M \sim C(P_1(t(t-1))) \oplus \dots \oplus C(P_p(t(t-1))).$$

Using Corollary 8 and the reconstruction principle of Section 2.1, we deduce that M is a $(1, 0, 0, 0)$ -quadratic sum.

- Conversely, assume that $M = A + B$ for some pair $(A, B) \in M_n(\mathbb{K})^2$ such that $A^2 = A$ and $B^2 = 0$. Assume furthermore that M has no eigenvalue in $\{0, 1\}$. This last assumption yields

$$\text{Ker } A \cap \text{Ker } B = \text{Ker}(A - I_n) \cap \text{Ker } B = \{0\}.$$

Therefore

$$\dim \text{Ker } A \leq n - \dim \text{Ker } B = \text{rk } B \quad \text{and} \quad \dim \text{Ker}(A - I_n) \leq \text{rk } B.$$

Adding these inequalities yields $n \leq 2 \text{rk } B$. However $2 \text{rk } B \leq \text{rk } B + \dim \text{Ker } B = n$ since $\text{Im } B \subset \text{Ker } B$. It follows that

$$\dim \text{Ker } A = \dim \text{Ker}(A - I_n) = \dim \text{Ker } B = \text{rk } B = \frac{n}{2}$$

and hence

$$\mathbb{K}^n = \text{Ker } A \oplus \text{Ker } B.$$

Set now $p := \frac{n}{2}$. Using a basis of \mathbb{K}^{2p} which is adapted to the decomposition $E = \text{Ker } B \oplus \text{Ker } A$, we find $P \in \text{GL}_n(\mathbb{K})$ and matrices C, D in $M_p(\mathbb{K})$ such that

$$A = P \begin{bmatrix} I_p & 0_p \\ C & 0_p \end{bmatrix} P^{-1} \quad \text{and} \quad B = P \begin{bmatrix} 0_p & D \\ 0_p & 0_p \end{bmatrix} P^{-1}.$$

Using $\text{Ker}(A - I_n) \cap \text{Ker } B = \{0\}$, we find that C is non-singular. Setting $Q := \begin{bmatrix} I_p & 0 \\ 0 & C \end{bmatrix}$, we finally find some $D' \in M_p(\mathbb{K})$ such that

$$M = (PQ) \begin{bmatrix} I_p & D' \\ I_p & 0_p \end{bmatrix} (PQ)^{-1} \sim U_{D'}.$$

The rational canonical form of D' yields monic polynomials P_1, \dots, P_q such that $D' \sim C(P_1) \oplus \dots \oplus C(P_q)$ and P_k divides P_{k+1} for every $k \in \{1, \dots, q-1\}$. By Lemmas 5 and 6, this yields

$$M \sim U_{C(P_1)} \oplus \dots \oplus U_{C(P_q)}.$$

Using Corollary 8, it follows that

$$M \sim C(P_1(t(t-1))) \oplus \cdots \oplus C(P_q(t(t-1))).$$

Finally, $P_k(t(t-1))$ divides $P_{k+1}(t(t-1))$ for every $k \in \{1, \dots, q-1\}$, and hence $P_1(t(t-1)), \dots, P_q(t(t-1))$ are the invariant factors of M . Since M has no eigenvalue in $\{0, 1\}$, we conclude that M satisfies condition (iii) in Theorem 1.

We conclude that equivalence (i) \Leftrightarrow (iii) of Theorem 1 holds for any square matrix with no eigenvalue in $\{0, 1\}$.

4 The case M is triangularizable with eigenvalues in $\{0, 1\}$

4.1 A review of Wang's results

In [5, Lemma 2.3], Wang proved the following characterization of pairs of nilpotent matrices (M, N) for which the sequences $(n_k(M, 0))_{k \geq 1}$ and $(n_k(N, 0))_{k \geq 1}$ are p -intertwined (generalizing a famous theorem of Flanders [2]).

Theorem 9 (Wang). *Let $p \in \mathbb{N}^*$ and $(M, N) \in M_r(\mathbb{K}) \times M_s(\mathbb{K})$ be a pair of nilpotent matrices. The following conditions are equivalent:*

- (i) *The sequences $(n_k(M, 0))_{k \geq 1}$ and $(n_k(N, 0))_{k \geq 1}$ are p -intertwined.*
- (ii) *There is a pair $(X, Y) \in M_{r,s}(\mathbb{K}) \times M_{s,r}(\mathbb{K})$ such that $M^p = XY$, $N^p = YX$, $MX = XN$ and $YM = NY$.*

Wang only considered the field of complex numbers but an inspection of his proof reveals that it holds for an arbitrary field.

In [5], implication (i) \Rightarrow (ii) of Theorem 9 is used, with $p = 2$, to obtain the following result:

Proposition 10. *Let $M \in M_n(\mathbb{K})$ be a triangularizable matrix with eigenvalues in $\{0, 1\}$ and assume that the sequences $(n_k(M, 0))_{k \geq 1}$ and $(n_k(M, 1))_{k \geq 1}$ are 2-intertwined. Then M is a $(1, 0, 0, 0)$ -quadratic sum.*

Again, Wang's proof [5, Lemma 2.2, "Sufficiency" paragraph] holds for an arbitrary field and we shall not reproduce it. We deduce that implication (iii) \Rightarrow (i) in Theorem 1 holds when M is triangularizable with eigenvalues in $\{0, 1\}$.

4.2 A necessary condition for being a $(1, 0, 0, 0)$ -quadratic sum

Here, we prove the converse of Proposition 10:

Proposition 11. *Let $M \in M_n(\mathbb{K})$ be a triangularizable matrix with eigenvalues in $\{0, 1\}$. Assume that M is a $(1, 0, 0, 0)$ -quadratic sum. Then the sequences $(n_k(M, 0))_{k \geq 1}$ and $(n_k(M, 1))_{k \geq 1}$ are 2-intertwined.*

Proving this will complete our proof of Theorem 1.

In [5], Wang proved Proposition 11 in the special case $\mathbb{K} = \mathbb{C}$. An inspection shows that his proof works for an arbitrary field of characteristic not 2, but fails for a field of characteristic 2 (due to Wang's systematic use of the division by 2). Our aim is to give a proof that works regardless of the characteristic of \mathbb{K} . In order to do this, we will reduce the situation to the one where no Jordan block of M has a size greater than 3 (in other words $M^3(M - I_n)^3 = 0$). Let us start by considering that special case:

Lemma 12. *Let $M \in M_n(\mathbb{K})$ be a $(1, 0, 0, 0)$ -quadratic sum such that $M^3(M - I_n)^3 = 0$. Then $n_3(M, 0) \leq n_1(M, 1)$ and $n_3(M, 1) \leq n_1(M, 0)$.*

Proof. We lose no generality in assuming that

$$M = \begin{bmatrix} I_p + N & 0 \\ 0 & N' \end{bmatrix},$$

where $p + q = n$, $(N, N') \in M_p(\mathbb{K}) \times M_q(\mathbb{K})$, and $N^3 = 0$ and $(N')^3 = 0$.

With the same block sizes, we may find some $B = \begin{bmatrix} B_1 & B_3 \\ B_2 & B_4 \end{bmatrix} \in M_n(\mathbb{K})$ such

that $B^2 = 0$ and $(M - B)^2 = M - B$. By Corollary 4, B commutes with $M(M - I_n) = \begin{bmatrix} N^2 + N & 0 \\ 0 & (N')^2 - N' \end{bmatrix}$. It follows that B_1 commutes with $N + N^2$,

whilst B_4 commutes with $N' - (N')^2$.

However $N = (N + N^2) - (N + N^2)^2$ and $N' = (N' - (N')^2) + (N' - (N')^2)^2$.

Therefore B_1 commutes with N , and B_4 commutes with N' .

Next, the identities $(M - B)^2 = M - B$ and $B^2 = 0$ yield:

$$M^2 - MB - BM = M - B.$$

We deduce:

$$N'B_2 + B_2N = 0 \quad ; \quad NB_3 + B_3N' = 0,$$

$$N^2 + N = NB_1 + B_1N + B_1 = (2N + I_p)B_1 \quad \text{and} \quad (N')^2 - N' = (2N' - I_q)B_4.$$

Therefore

$$B_1 = (I_p + 2N)^{-1}(N + N^2) = (I_p - 2N + 4N^2)(N + N^2) = N - N^2$$

and

$$B_4 = (I_q - 2N')^{-1}(N' - (N')^2) = (I_q + 2N' + 4(N')^2)(N' - (N')^2) = N' + (N')^2.$$

Using this, we compute

$$B^2 = \begin{bmatrix} N^2 + B_3B_2 & ? \\ ? & (N')^2 + B_2B_3 \end{bmatrix}.$$

Since $B^2 = 0$, we deduce that

$$N^2 = (-B_3)B_2 \quad \text{and} \quad (-N')^2 = B_2(-B_3).$$

Recalling that

$$(-N')B_2 = B_2N \quad \text{and} \quad N(-B_3) = (-B_3)(-N'),$$

Theorem 9 yields $n_3(N, 0) \leq n_1(-N', 0)$ and $n_3(-N', 0) \leq n_1(N, 0)$, i.e., $n_3(M, 1) \leq n_1(M, 0)$ and $n_3(M, 0) \leq n_1(M, 1)$. \square

We finish by deducing the general case from the above special one:

Proof of Proposition 11. We think in terms of endomorphisms of the space \mathbb{K}^n . Let u be an endomorphism of \mathbb{K}^n such that $u^n(u - \text{id})^n = 0$, and assume that there is an idempotent endomorphism a and a square-zero endomorphism b such that $u = a + b$.

By Corollary 4, $E_k := \text{Ker}(u^k(u - \text{id})^k)$ is stabilized by a and b for every $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then a , b and u induce endomorphisms a' , b' and u' of E_{k+3}/E_k , with $(a')^2 = a'$, $(b')^2 = 0$, and $(u')^3(u' - \text{id})^3 = 0$ (as $u^3(u - \text{id})^3$ maps E_{k+3} into E_k). Applying Lemma 12 to u' , we find that $n_3(u', 1) \leq n_1(u', 0)$ and $n_3(u', 0) \leq n_1(u', 1)$. In order to conclude, it suffices to note that

$$\forall i \in \{1, 2, 3\}, \quad n_i(u', 0) = n_{k+i}(u, 0) \quad \text{and} \quad n_i(u', 1) = n_{k+i}(u, 1).$$

Note indeed, using the kernel decomposition theorem, that the characteristic subspace of u' for the eigenvalue 0 is $(\text{Ker } u^{k+3} \oplus \text{Ker}(u - \text{id})^k) / (\text{Ker } u^k \oplus \text{Ker}(u -$

$\text{id})^k$), and hence the nilpotent part of u' is similar to the endomorphism $v : x \mapsto u(x)$ of $\text{Ker } u^{k+3} / \text{Ker } u^k$. However $\text{Ker } v^i = \text{Ker } u^{k+i} / \text{Ker } u^k$ for every $i \in \{0, 1, 2, 3\}$. Therefore

$$\begin{aligned} n_i(u', 0) = n_i(v, 0) &= (\dim \text{Ker } u^{k+i} - \dim \text{Ker } u^k) - (\dim \text{Ker } u^{k+i-1} - \dim \text{Ker } u^k) \\ &= n_{k+i}(u, 0) \end{aligned}$$

for every $i \in \{1, 2, 3\}$. In the same way, one proves that $n_i(u', 1) = n_{k+i}(u, 1)$ for every $i \in \{1, 2, 3\}$.

The special cases $i = 1$ and $i = 3$ yield $n_{k+3}(u, 1) \leq n_{k+1}(u, 0)$ and $n_{k+3}(u, 0) \leq n_{k+1}(u, 1)$. \square

This completes our proof of Theorem 1.

5 Addendum : a simplified proof of a result on linear combinations of idempotent matrices

In this last section, we wish to show how the strategy of Section 4.2 may be adapted so as to yield a simplified proof of the following result of [4]:

Proposition 13. *Let α, β be distinct elements of $\mathbb{K} \setminus \{0\}$. Let $M \in \text{M}_n(\mathbb{K})$ be an $(\alpha, 0, \beta, 0)$ -quadratic sum such that $(M - \alpha I_n)^n (M - \beta I_n)^n = 0$. Then the sequences $(n_k(M, \alpha))_{k \geq 1}$ and $(n_k(M, \beta))_{k \geq 1}$ are 1-intertwined.*

Proof. As in the proof of Proposition 11, one can use the commutation with $(M - \alpha I_n)(M - \beta I_n) = M(M - (\alpha + \beta)I_n) + \alpha\beta I_n$ (see Lemma 3) to reduce the situation to the one where $(M - \alpha I_n)^2 (M - \beta I_n)^2 = 0$. In that case, we lose no generality in assuming that

$$M = (\alpha I_p + N) \oplus (\beta I_q + N'),$$

where $p + q = n$, $N \in \text{M}_p(\mathbb{K})$ and $N' \in \text{M}_q(\mathbb{K})$ satisfy $N^2 = 0$ and $(N')^2 = 0$. Note that

$$(M - \alpha I_n)(M - \beta I_n) = (\alpha - \beta)(N \oplus (-N')).$$

Let then A and B be idempotent matrices such that $M = \alpha A + \beta B$. Split

$$A = \begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix},$$

where A_1, A_2, A_3, A_4 are respectively $p \times p$, $q \times p$, $p \times q$ and $q \times q$ matrices. By Lemma 3, A commutes with $(M - \alpha I_n)(M - \beta I_n)$; as $\alpha \neq \beta$, we deduce that A_1 commutes with N .

On the other hand, the identity $(M - \alpha A)^2 = \beta(M - \alpha A)$ yields:

$$\alpha(\alpha + \beta)A = \alpha(AM + MA) + \beta M - M^2.$$

Evaluating the upper-left blocks on both sides and using the commutation $A_1 N = N A_1$, we deduce:

$$\alpha(\alpha + \beta)A_1 = 2\alpha(\alpha I_n + N)A_1 + \beta(\alpha I_n + N) - (\alpha I_n + N)^2$$

and hence

$$\alpha((\beta - \alpha)I_n - 2N)A_1 = \alpha(\beta - \alpha)I_n + (\beta - 2\alpha)N.$$

As $\alpha(\beta - \alpha) \neq 0$ and $N^2 = 0$, we deduce that

$$A_1 = \left(I_n + \frac{\beta - 2\alpha}{\alpha(\beta - \alpha)}N\right)\left(I_n - \frac{2}{\beta - \alpha}N\right)^{-1} = I_n + \frac{\beta}{\alpha(\beta - \alpha)}N,$$

and it follows that the upper-left block of B is $\frac{1}{\beta}(\alpha I_n + N - \alpha A_1) = \frac{\alpha}{\beta(\alpha - \beta)}N$.

By symmetry, one has $A_4 = \frac{\beta}{\alpha(\beta - \alpha)}N'$. We deduce that

$$A - A^2 = \begin{bmatrix} \frac{\beta}{\alpha(\alpha - \beta)}N - A_3 A_2 & ? \\ ? & \frac{\beta}{\alpha(\beta - \alpha)}N' - A_2 A_3 \end{bmatrix}.$$

Setting $X := \alpha(\alpha - \beta)A_3$ and $Y := \frac{1}{\beta}A_2$, we find:

$$N = XY \quad \text{and} \quad -N' = YX.$$

The main theorem of [2] (or Theorem 9 for $p = 1$, noting that $NX = XYX = X(-N')$ and $YN = YXY = (-N')Y$) then shows that the sequences $(n_k(N, 0))_{k \geq 1}$ and $(n_k(-N', 0))_{k \geq 1}$ are 1-intertwined, i.e., the sequences $(n_k(M, \alpha))_{k \geq 1}$ and $(n_k(M, \beta))_{k \geq 1}$ are 1-intertwined. \square

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